Daniel Müllner Spring 2012

Applied Algebraic Topology, Homework 4

Due: Tuesday, May 29 in class

1. Persistent homology of the pinched torus

The goal of this exercise is to build a model for the pinched torus from problem 3 i) in Homework 2 and compute its persistent homology. It is allowed to use material from the javaPlex tutorial for this exercise, but please explain what you do and what your (modified) MATLAB code does. Print your MATLAB code out, along with the figures that you generated with it.

Start with the "flat" torus from Exercise 4 in the javaPlex tutorial: Let T be the quotient space of the unit square $[0,1] \times [0,1]$, where the left and the right edge are identified, and likewise the top and bottom edge. If $x = (x_1, x_2)$ denotes the representative of a point on T, a metric on T is defined by

$$d_T(x,y) := \sqrt{\min(|x_1 - y_1|, 1 - |x_1 - y_1|)^2 + \min(|x_2 - y_2|, 1 - |x_2 - y_2|)^2}$$

This is the metric which is induced on T by the Euclidean distance on the unit square, where we allow shortest paths to pass through the edges which are glued together.

For the pinched torus, we consider a particular meridian L_1 on T, namely the points $(.5, x_2)$ for $x_2 \in [0,1]$. Define the following function on the quotient space $P := T/L_1$, which measures the distance from a point to the meridian L_1:

$$l : P \to \mathbb{R}$$
$$[x_1, x_2] \mapsto |x_1 - .5|$$

Now define a metric on $P := T/L_1$ as follows: d_P is the function $P \times P \to \mathbb{R}$ given by

$$([x],[y]) \mapsto \min(d_T(x,y), l(x) + l(y)).$$

i) Check that l is well-defined and prove that d_p defines a metric, i. e. it fulfills the following axioms:

1. Positive definiteness: $d_P([x],[y]) \geq 0$, and $d_P([x],[y]) = 0$ if and only if $[x] = [y]$ in P
2. Symmetry: $d_P([x],[y]) = d_P([y],[x])$ for all $[x],[y] \in P$
3. Triangle inequality: $d_P([x],[z]) \leq d_P([x],[y]) + d_P([y],[z])$ for all $[x],[y],[z] \in P$.

You may assume the analogous properties for d_T without proof.

ii) Modify the MATLAB function `flatTorusDistanceMatrix` from the javaPlex tutorial so that it generates a matrix of pairwise distances measured by d_P instead of d_T.

iii) Build the distance matrix for 1000 random points on the pinched torus P. Build a lazy witness stream from this matrix with 50 landmarks chosen via sequential max-min and with $\nu = 1$. The maximum filtration value needs to be chosen experimentally so that javaPlex does not abort with a memory error, but large enough to reveal good persistence information. What persistent homology diagrams do you get in degrees 0, 1 and 2? Which homology groups do you expect from the ideal space P?

Next, repeat the experiment with random landmarks several times. Show the "best" and "worst" barcode that you get (according to your judgment). Why do the results change quite much from one instance to another?

iv) If you choose the maximum filtration value in the witness complex (say, with sequential max-min sampling) large enough, you will notice that the diagram for H_2 has several small bars towards the larger end. Give an explanation for these bars.

(30 points)