Daniel Müllner Spring 2012

Applied Algebraic Topology, Homework 2

Due: Tuesday, May 15 in class

1. Prove the following lemma:

Lemma. *Consider the following commutative diagram of modules over a commutative ring R:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0
\end{array}$$

Suppose that the rows are exact and the two vertical maps f and h are isomorphisms. Then g is also an isomorphism.

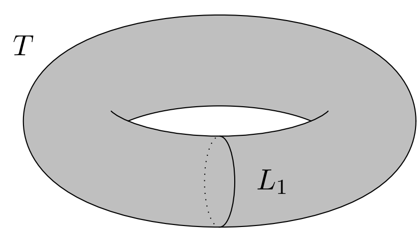
(10 points)

This is a version of the so-called **five lemma**. You'll easily find proofs of this fundamental lemma in the internet. Try to prove it yourself! If it helps you conceptually, prove the five lemma for vector spaces only. Chances are high that your arguments carry over word by word to the general case of R-modules.

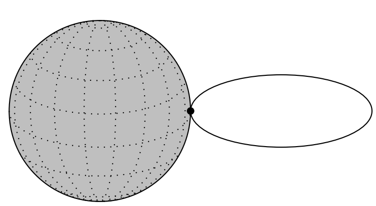
2. Let K be the Möbius strip, and let L be its "edge". Compute $H_*(K;\mathbb{Z})$ and $H_*(K,L;\mathbb{Z})$. (Fix a triangulation of K, if you like, or represent the Möbius strip in any other way.)

(8 points)

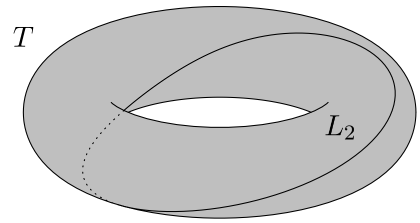
3. i) Let T be the 2-torus and L_1 be a loop on T that wraps once around the meridian, as in the following figure:



Prove that the quotient space T/L_1 is homotopy equivalent to the union of a 2-sphere and a 1-sphere along a single point. (This is denoted $S^2 \vee S^1$, S^2 "wedge" S^1.)



ii) Now let L_2 be a loop on T that wraps once around the equator and once around the meridian:

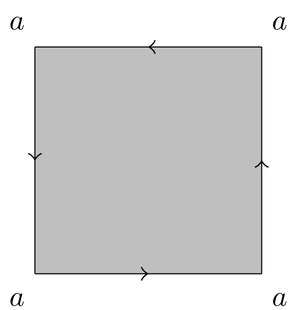


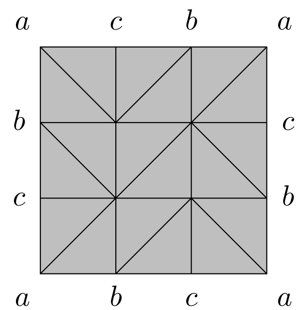
Prove that the quotient space T/L_2 is again homotopy equivalent to $S^2 \vee S^1$.

Pictorial proofs are permitted!

(10 points)

4. Consider the quotient space K of a square, where all four boundary edges are identified to a single circle, with orientations as given below. A sample triangulation is given for clarity, which you are free to use or ignore.





Compute the homology groups $H_*(K;\mathbb{Z}/2)$, $H_*(K;\mathbb{Z}/3)$ and $H_*(K;\mathbb{Z})$. Tip: Remove the small inner square.

(10 points)