Daniel Müllner Spring 2012

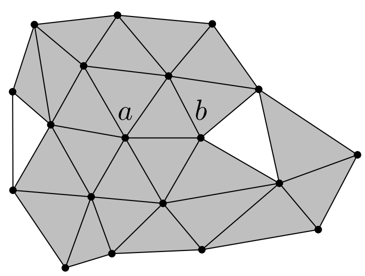
Applied Algebraic Topology, Homework 1

Due: Tuesday, May 1 in class

1. Given a simplicial complex K, the **star** $\mathrm{St}(\sigma)$ of a simplex $\sigma \in K$ is the collection of its cofaces. It is in general not a simplicial complex in its own right. The simplicial complex consisting of $\mathrm{St}(\sigma)$ and all faces is called the **closed star** $\overline{\mathrm{St}}(\sigma)$. The **link** $\mathrm{Lk}(\sigma)$ is the collection of simplices in $\overline{\mathrm{St}}(\sigma)$ which are disjoint from σ:

$$\mathrm{Lk}(\sigma) := \{\tau \in \overline{\mathrm{St}}(\sigma) \mid \sigma \cap \tau = \emptyset\}.$$

i) What are the stars and links of the simplices (a), (b) and (ab) in the following complex?



ii) Prove or give a counterexample: $\mathrm{Lk}(\sigma)$ is always a simplicial complex.

iii) Prove or give a counterexample: For every edge (ab) in a simplicial complex, $\mathrm{Lk}(ab) = \mathrm{Lk}(a) \cap \mathrm{Lk}(b)$.

(6 points)

2. Recall that the simplicial complex made out of a d-dimensional simplex Δ^d and all its faces has $2^{d+1} - 1$ simplices.

i) Find a formula for the number of simplices in the first barycentric subdivision $\mathrm{Sd}(\Delta^d)$. A recursive formula with proof is fine.

ii) Prove that the number of simplices in $\mathrm{Sd}(\Delta^d)$ is bounded below by $(d+1)!$.

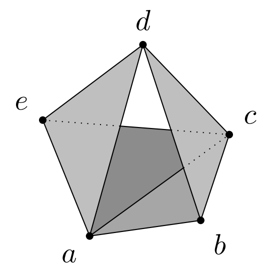
(10 points)

3. Write a MATLAB function to determine the orientation of a simplex. The function should receive a vector with a permutation $(i_1, \ldots, i_k)$ of $(1, \ldots, k)$ as the argument and return a boolean value that states whether the abstract $(k-1)$-simplex $\sigma(i_1, \ldots, i_k)$ has the same orientation as $\sigma(1, \ldots, k)$. Design a function with asympotic runtime complexity $O(k)$.

Test the function with the sequences $(1, 3, 2)$, $(4, 5, 3, 2, 1)$, $(6, 7, 3, 5, 4, 1, 8, 2)$ and $(2, 7, 4, 1, 6, 3, 8, 5)$ and report the results.

Send your code to Daniel by e-mail for credit. *(10 points)*

4. Consider the following simplicial complex K. Fix $\mathbb{R}$ as the coefficient field.



i) What are the dimensions of $C_k(K)$ for $k = 0, 1, 2$?

ii) Choose bases for $C_k(K)$. Give ∂_1 and ∂_2 in matrix form with respect to these bases.

iii) Determine the rank and nullity of ∂_1, ∂_2.

iv) Give bases for the cycle and boundary vector spaces $Z_2(K)$, $B_2(K)$, $Z_1(K)$, $B_1(K)$, $B_0(K)$.

v) Compute the homology of K.

(10 points)