Orientation reversal of manifolds

Summary of my PhD thesis
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Manifolds which do not allow orientation-reversing maps to themselves have been known for a long time. Textbook examples are the complex projective spaces $\mathbb{C}P^{2n}$ in dimensions that are divisible by 4 and some lens spaces in dimensions congruent 3 modulo 4. Apart from these well-known examples, only sparse results were known (see below for some of them), so my PhD project is dedicated to the following question:

When does a manifold *not* admit an orientation-reversing map to itself?

In the following, if not otherwise stated, we deal always with closed, connected, orientable, smooth manifolds. The question whether orientation reversal is possible can be asked in various categories, as one can require the map of degree −1 to be a homeomorphism or diffeomorphism. The following figure lists the categories that were considered, as well as the interrelations between them.

or. rev. diffeomorphism	$\Rightarrow$	or. rev. homeomorphism	$\Rightarrow$	or. rev. homotopy equivalence	$\Rightarrow$	self map of degree −1
$\Downarrow$					$\Leftrightarrow$	
bordism $M \sim -M$					for 1-conn. manifolds	

Note that the bordism question was solved completely (the final step is due to Wall [6]): A manifold is oriented bordant to its negative if and only if all its Pontrjagin numbers vanish.

A manifold is called *chiral* or *amphicheiral* according to whether the orientation cannot or can be reversed by a self-map (in one of the above categories and to be specified in each context). In fact, instances are known where these categories differ: some lens spaces are amphicheiral by a homotopy equivalence but not by a homeomorphism, and some exotic homotopy spheres are "topologically amphicheiral" but "smoothly chiral", i. e. there exists an orientation-reversing homeomorphism but no diffeomorphism. The terms "chiral" and "amphicheiral" are used because of parallels between the topology of 3-dimensional manifolds and knot theory, where these notions already exist.

In low dimensions, the following facts are easily available: in dimension zero, a single point is chiral. In dimension 1 and 2, all (closed, orientable, ...) manifolds are smoothly amphicheiral. One can choose, e. g., the reflection at the equator of the circle or the "equatorial" plane in an appropriate embedding of an oriented surface in $\mathbb{R}^3$ as an orientation-reversing involution.

Own results

Previously, it was not known whether there exist chiral manifolds in every dimension ≥ 3. This problem could be solved, for chirality in the strongest possible sense:

Theorem 1. *In every dimension at least* 3*, there exists a closed, connected, orientable, smooth manifold that does not admit a continuous map to itself with degree* -1.

The proof was done in two steps:

1. I constructed a series of examples in every odd dimension ≥ 3, thereby identifying a new obstruction to orientation reversal.

2. In even dimensions I used that sometimes cartesian products of chiral manifolds are again chiral.

The examples in odd dimensions are all aspherical manifolds, i. e. Eilenberg-MacLane spaces. Given a self-map of M, the effect on homology thus depends only on the endomorphism of the fundamental group. The manifolds M^n which were constructed have fundamental groups such that no endomorphism of $\pi_1(M)$ induces $-\mathrm{id}$ on $H_n(M) \cong \mathbb{Z}$.

Since this construction relies so obviously on the fundamental group, the next step is to ask for *simply-connected* examples in dimension ≥ 3. For these manifolds, complete classification results are available up to dimension 6 (the work of Perelman [4], Freedman [3], Barden [1] and Zhubr [7]). One can deduce from the classification theorems that every smooth, simply-connected manifold is smoothly amphicheiral in dimensions 3, 5 and 6. In dimension 4, every simply-connected manifold is topologically amphicheiral if and only if its signature is zero.

For every higher dimension, the existence of simply-connected, chiral manifolds could be proved.

Theorem 2. *In every dimension at least* 7*, there exists a closed, simply-connected, orientable, smooth manifold that does not admit a continuous map to itself with degree* -1.

I could use fiber bundles with fiber S^{k-1} over S^k and again products of existing examples to cover all dimensions except 9, 10, 13 and 17. Since for these remaining four dimensions, the aim still was to find chiral manifolds in the strongest possible sense (i. e. without orientation reversing homotopy equivalences), I tried mainly techniques which use the Postnikov tower in some way.

In general, the task of producing new chiral manifolds is twofold:

1. *Find a new mechanism/obstruction to orientation reversal.* With the Postnikov tower, this means: Construct an appropriate finite tower of principal $K(\pi, n)$-fibrations and fix an element in the integral homology of one of the stages that is to be the image of the fundamental class of the manifold. Then prove that (by the mechanism that lies in the particular

construction) this homology class can never be mapped to its negative under any self-map of a single Postnikov stage or of the partial Postnikov tower.

2. *Show that this obstruction can be realized by a manifold.* In the case of the Postnikov tower, one must prove that there is indeed a manifold with the correct partial homotopy type and the correct image of the fundamental class in the Postnikov approximation. This step involves bordism computations and surgery techniques.

Truly new mechanisms for chirality had to be found in dimensions 9 and 10; dimensions 13 and 17 could then be handled by products of the new examples and other chiral manifolds. The proof that a certain homology class is never mapped to its negative under any self-map of a Postnikov stage/the partial Postnikov tower relied on mod-3 Steenrod operations in dimension 10 and a mixture of rational homotopy theory and additional integral information in dimension 9.

In order to further shed light on the phenomenon of chirality, I considered the question which manifolds are bordant to a chiral one. Apart from dimension 1 and 2, where there are no chiral manifolds, every oriented bordism class contains chiral manifolds.

Theorem 3. *In every dimension ≥ 3, every closed, smooth, oriented manifold is oriented bordant to a manifold of this type which is connected and homotopically chiral.*

For proving theorem 3, previously constructed examples could be used or extended except for one case: a chiral 4-dimensional manifold with signature zero. Here, I exhibited finite groups π that have no outer automorphisms and such that $H_4(\pi)$ contains elements of order > 2. The smallest possible group of this type is the product $G_3 \times G_7$, where G_p is a semidirect product of cyclic groups $\mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)$ and $\mathbb{Z}/(p-1)$ is identified with $\mathrm{Aut}(\mathbb{Z}/p)$.

Since inner automorphisms act trivially on group homology, it is then sufficient to prove the existence of a 4-dimensional manifold M with fundamental group π, signature 0 and the correct image of the fundamental class under a 2-equivalence $M \to K(\pi, 1)$.

The results so far have produced a number of chiral manifolds with various constraints. Aiming in the opposite direction, it is also interesting to prove amphicheirality of manifolds in nontrivial circumstances. A very interesting class of manifolds in this context are products of lens spaces. For single lens spaces, it is known which are amphicheiral by a diffeomorphism and by a homotopy equivalence. Furthermore, it follows from more general statements in the thesis about products of chiral manifolds that products of lens spaces of *different* dimensions are homotopically chiral whenever each single factor is homotopically chiral:

Example 4. *Let $L := L_1 \times \ldots \times L_k$ be a product of lens spaces of pairwise different dimensions. Then L is homotopically chiral if and only if this holds for each single factor.*

This leads to the question what happens with lens spaces of the same dimension. I showed, using the modified surgery theory of Kreck, that the product of two lens spaces can be smoothly amphicheiral, also in non-obvious cases (which are: one of the factors is amphicheiral or the factors are diffeomorphic).

Theorem 5. *Let r_1 and r_2 be coprime odd integers and let L_1 and L_2 be (any) 3-dimensional lens spaces with fundamental groups $\mathbb{Z}/r_1$ resp. $\mathbb{Z}/r_2$. Then the product $L_1 \times L_2$ is smoothly amphicheiral.*

Conclusion

In chemistry, a molecule is called chiral if it cannot be superimposed on its mirror image [5]. Another definition which captures the properties of flexible and topologically complex molecules better is given by [2]: A molecule "that can chemically change itself into its mirror image" is called achiral and chiral if it cannot. Chiral molecules have the same physical properties like melting and boiling points but they behave optically and chemically differently. With this analogy in mind, it seems a very natural question to ask whether an orientable manifold with its two orientations yields "the same" or "different" objects. Although the manifolds one usually imagines (spheres and 2-dimensional surfaces) are amphicheiral, I could show that chiral manifolds exist in every dimension greater than two. Furthermore, my results give a little insight into the variety of mechanisms that can obstruct orientation reversal in the homotopy type. Aiming in the opposite direction, I showed that products of lens spaces can have orientation-reversing diffeomorphisms in nontrivial circumstances.

References

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